APPENDIX I: IDENTITIES INVOLVING THE CHRISTOFFEL SYMBOLS

The Christoffel symbols of the second kind are defined as

$$\Gamma^{i}_{jk} = \frac{1}{2} g^{il} \left[\frac{\partial g_{jl}}{\partial u_k} + \frac{\partial g_{kl}}{\partial u_j} - \frac{\partial g_{jk}}{\partial u_l} \right] \tag{1}$$

The derivative of the covariant metric tensor expressed in terms of the Christoffel symbols of the second kind is:

$$\frac{\partial g_{mn}}{\partial u^k} = + \left[g_{mu} \Gamma^{u}_{nk} + g_{nu} \Gamma^{u}_{mk} \right] \tag{2}$$

The derivative of the contravariant metric tensor expressed in terms of the Christoffel symbols of the second kind is:

$$\frac{\partial g^{mn}}{\partial u^k} = - \left[g^{mu} \Gamma^{n}_{uk} + g^{nu} \Gamma^{m}_{uk} \right] \tag{3}$$

GRAM MATRIX

The identity for the differentiation of the elements of an inverse matrix with respect to the elements of the corresponding original matrix from [2] (page 18):

$$\frac{\partial h^{pq}}{\partial h_{rs}} = - h^{ps} h^{rq} \tag{4}$$

The derivative of the Gram matrix with respect to the coordinates:

$$\frac{\partial h_{pq}}{\partial u^k} = g^{mn} \left[(H^{p}_{mk} - \Gamma^{u}_{mk} H^{p}_{u}) H^{q}_{n} + (H^{q}_{mk} - \Gamma^{u}_{mk} H^{q}_{u}) H^{p}_{n} \right] \tag{5}$$

The derivative of the inverse Gram matrix with respect to the coordinates:

$$\frac{\partial h^{pq}}{\partial u^k} = - h^{pr} h^{sq} g^{mn} \left[(H^{r}_{mk} - \Gamma^{u}_{mk} H^{r}_{u}) H^{s}_{n} + (H^{s}_{mk} - \Gamma^{u}_{mk} H^{s}_{u}) H^{r}_{n} \right] \tag{6}$$

CHRISTOFFEL SYMBOLS TRANSFORMATION

Under a change of coordinates from u^i to v^i the Christoffel symbols transform according to the equation

$$\Gamma^{i}_{jk} = (\Gamma^{l}_{mn} X^{i}_{l}) X^{m}_{j} X^{n}_{k} + X^{i}_{n} X^{n}_{jk} \tag{7}$$

where we have written the following shorthand notation

$$X_j^i = \frac{\partial u^i}{\partial v^j} \qquad (8)$$

$$X_{jk}^i = \frac{\partial^2 u^i}{\partial v^j \partial v^k} \qquad (9)$$

INVERSE MATRIX DIFFERENTIATION

Consider the definition of the inverse matrix

$$a_{ik} a^{kj} = \delta_j^i \qquad (10)$$

Differentiating with respect to u^i and rearranging indices we obtain

$$\frac{\partial a^{ij}}{\partial u^k} = -a^{im} \frac{\partial a_{mn}}{\partial u^k} a^{nj} \qquad (11)$$

In matrix notation

$$\frac{\partial A^{-1}}{\partial u^k} = -A^{-1} \frac{\partial A}{\partial u^k} A^{-1} \qquad (12)$$

APPENDIX II: CONSTRUCTION OF THE CONSTRAINED GEODESIC

HOLONOMIC CONSTRAINTS

Here we derive the solution of the problem of shortest path subject to a number of algebraic equality constraints.

The problem can be stated as the variational integral of minimum arclength

$$s = \int_0^{t_0} (g_{kl} u_t^k u_t^l)^{\frac{1}{2}} dt \tag{1}$$

subject to the algebraic equality constraints

$$H^p(u) = 0 \qquad (p = 1 \cdots m) \tag{2}$$

This type of problem is solved using the method of Lagrange multipliers [4] by augmenting the minimization functional with a linear combination of the constraint equations using certain Lagrange multipliers λ_p

$$F(u, u_t) = (g_{kl} u_t^k u_t^l)^{\frac{1}{2}} + \sum \lambda_p H^p(u) \tag{3}$$

The Euler-Lagrange equation for this type of functional is

$$\frac{\partial F}{\partial u^i} - \frac{d}{dt}\frac{\partial F}{\partial u_t^i} = 0 \tag{4}$$

The first term in the Euler-Lagrange equation (4) is computed as follows

$$\frac{\partial F}{\partial u^i} = \frac{1}{2s'}\frac{\partial g_{kl}}{\partial u^i} u_t^k u_t^l + \sum \lambda_p H_i^p \tag{5}$$

where we have used the abbreviation

$$s' = \frac{ds}{dt} = (g_{kl} u_t^k u_t^l)^{\frac{1}{2}} \tag{6}$$

The second term in the Euler-Lagrange equation (4) is computed as follows

$$\frac{\partial F}{\partial u_t^i} = \frac{1}{2s'} 2 g_{ik} u_t^k \tag{7}$$

from which by differentiation with respect to the parameter t

$$\frac{d}{dt}\frac{\partial F}{\partial u_t^i} = \frac{1}{s'}\left[g_{ik}u_{tt}^k + \frac{\partial g_{ik}}{\partial u^l}u_t^k u_t^l\right] - \frac{s''}{(s')^2}g_{ik}u_t^k \tag{8}$$

From here on we may take the parameter **t** to be the arclength **s**, so that we have

$$\begin{aligned} s' &= 1 \\ s'' &= 0 \end{aligned} \tag{9}$$

Substituting (5) and (8) into the Euler-Lagrange equation (4) and invoking (9) we obtain

$$g_{ik}u_{ss}^k - \frac{1}{2}\left[\frac{\partial g_{kl}}{\partial u^i} - \frac{\partial g_{ik}}{\partial u^l} - \frac{\partial g_{il}}{\partial u^k}\right]u_s^k u_s^l = \sum \lambda_p H_i^p \tag{10}$$

The second term involving the derivatives of the metric tensor has been split into symmetric equal parts to permit introducing the Christoffel symbols of the second kind

$$\Gamma_{jk}^i = \frac{1}{2}g^{il}\left[\frac{\partial g_{jl}}{\partial u_k} + \frac{\partial g_{kl}}{\partial u_j} - \frac{\partial g_{jk}}{\partial u_l}\right] \tag{11}$$

Substituting the Christoffel symbols (11) into (10) we obtain after raising the index

$$u_{ss}^i + \Gamma_{kl}^i u_s^k u_s^l = \sum \lambda_q g^{ik} H_k^q \tag{12}$$

The Lagrange multipliers are determined by the requirement that the second derivatives of the constraint functions with respect to arclength vanish along the geodesic

$$H_{ss}^p = H_{kl}^p u_s^k u_s^l + H_k^p u_{ss}^k = 0 \tag{13}$$

Multiplying (12) by H_i^p and eliminating the terms containing second derivatives u_{ss}^i using (13), we obtain

$$(g^{kl}H_k^p H_l^q)\lambda_q = -(H_{kl}^p - \Gamma_{kl}^m H_m^p)u_s^k u_s^l \tag{14}$$

We introduce the symmetric Gram matrix of the constraint derivatives [the inner products of the normals to the constraint surfaces]

$$h_{pq} = g^{kl}H_k^p H_l^q \tag{15}$$

and define its inverse by the notation

$$h^{pr} h_{rq} = \delta^p_q \tag{16}$$

Multiplying equation (14) by the inverse of the Gram matrix, we obtain explicit expressions for the Lagrange multipliers

$$\lambda_p = - h^{pq} (H^q_{kl} - \Gamma^m_{kl} H^q_m) \, u^k_s u^l_s \tag{17}$$

Substituting the Lagrange multipliers back into equation (12), we obtain the equation of the constrained geodesic

$$u^i_{ss} + \Gamma^i_{kl} u^k_s u^l_s + h^{pq} (g^{im} H^p_m)(H^q_{kl} - \Gamma^m_{kl} H^q_m) \, u^k_s u^l_s = 0 \tag{18}$$

It is possible to write the constrained geodesic in a unified form which may be used to represent both the case of holonomic constraints and the case of a combination of holonomic and non-holonomic constraints

$$u^i_{ss} + \Gamma^i_{jk} u^j_s u^k_s = - h^{pq} \alpha_p \beta_q \tag{19}$$

where

$$\alpha_p = (g^{ik} H^p_k) \tag{20}$$

$$\beta_p = (H^p_{kl} - \Gamma^n_{kl} H^p_n) \, u^k_s u^l_s \tag{21}$$

NON-HOLONOMIC CONSTRAINTS

In the case of a combination of holonomic and non-holonomic constraints we have

$$H^p(u) = 0 \qquad (p = 1 \cdots m_1) \tag{22}$$

$$G^q(u, u_s) = 0 \qquad (q = 1 \cdots m_2) \tag{23}$$

The augmented objective function for the method of Lagrange Multipliers is in this case

$$F(u, u_t) = (g_{kl} \, u^k_t u^l_t)^{\frac{1}{2}} + \sum \lambda_p H^p(u) + \sum \mu_p G^p(u, u_t) \tag{24}$$

Substituting the functional (21) in the Euler-Lagrange equation (4) we obtain

$$u^i_{ss} + \Gamma^i_{kl} u^k_s u^l_s = \sum \lambda_p g^{ik} H^p_k + \sum \mu_p g^{ik} \left[\frac{\partial G^p}{\partial u^k} - \frac{d}{ds} \frac{\partial G^p}{\partial u^k_s} \right] \tag{25}$$

The Lagrange multipliers λ and μ are obtained from the conditions

$$H_{ss}^{p} = H_{kl}^{p} u_s^k u_s^l + H_k^p u_{ss}^k = 0 \qquad (p = 1 \cdots m_1) \tag{26}$$

$$G_s^{p} = G_k^p u_s^k + \frac{\partial G^p}{\partial u_s^k} u_{ss}^k = 0 \qquad (p = 1 \cdots m_2) \tag{27}$$

Combining the Lagrange multipliers for both types of constraints into a composite vector $\gamma = [\lambda, \mu]^T$ and substituting (22) into (23) and (24) we obtain the system of equations

$$h_{pq} \gamma_q = -\beta_p \tag{28}$$

The inner product matrix is now a partitioned matrix whose factors reflect the number of constraints of each type

$$h_{pq} = \begin{bmatrix} g^{kl} H_k^p H_l^q & g^{kl} H_k^p \left(G_l^q - \frac{d}{ds}\frac{\partial G^q}{\partial u_s^l}\right) \\ g^{kl} \frac{\partial G^p}{\partial u_s^k} H_l^q & g^{kl} \frac{\partial G^p}{\partial u_s^k} \left(G_l^q - \frac{d}{ds}\frac{\partial G^q}{\partial u_s^l}\right) \end{bmatrix} \tag{29}$$

The extended right hand side is found as

$$\beta_p = \begin{bmatrix} (H_{kl}^p - \Gamma_{kl}^n H_n^p) u_s^k u_s^l \\ G_k^p u_s^k - \Gamma_{kl}^n \frac{\partial G^p}{\partial u_s^n} u_s^k u_s^l \end{bmatrix} \tag{30}$$

Substituting the Lagrange multipliers obtained from the solution of (29) into (26) we obtain the constrained geodesic for the case of a combination of holonomic and non-holonomic constraints in the same form as for the case of holonomic constraints only

$$u_{ss}^i + \Gamma_{kl}^i u_s^k u_s^l = -h^{pq} \alpha^p \beta^q \tag{31}$$

The other extended vector is

$$\alpha_p = \begin{bmatrix} g^{il} H_l^p \\ g^{il} \left(G_l^p - \frac{d}{ds}\frac{\partial G^p}{\partial u_s^l}\right) \end{bmatrix} \tag{32}$$

This completes the construction for the non-holonomic case. Note that the quadratic form in the right hand side of (32) is of the form $[A^T H^{-1} B]$ and involves the inverse of the non-symmetric inner product matrix.

NOTES ON THE HOLONOMIC CASE

The constrained geodesic **C** has been constructed such that the second derivatives of the constraint equations with respect to arclength (13) vanish along **C** which is readily verified. The constraints (2) are therefore linear functions of arclength. If the initial direction for the constrained geodesic $u_s^i(0)$ is chosen to lie in the tangent plane of the manifold [causing the first derivatives of the constraints $H_s^p(0)$ with respect to arclength to be zero], then the constraints must be constant along the geodesic. Finally, if the starting point of the geodesic $u^i(0)$ is chosen on the manifold [causing the constraints $H^p(0)$ to be satisfied], then the constraints vanish everywhere along **C**.

For the parameter **s** to represent arclength along **C**, the following identity must hold for all s

$$g_{kl}\, u_s^k u_s^l = 1 \tag{33}$$

If we differentiate equation (19) with respect to s we obtain

$$\frac{d}{ds}\left(g_{ij}\, u_s^i u_s^j\right) = \frac{\partial g_{ij}}{\partial u^k}\, u_s^i u_s^j u_s^k + 2\, g_{ij}\, u_{ss}^i u_s^j = 0 \tag{34}$$

Expressing the derivatives of the metric tensor in terms of the Christoffel symbols (Appendix I) and eliminating the second derivative using the constraint geodesic equation we obtain

$$\left[g_{il}\Gamma^l_{jk} + g_{jl}\Gamma^l_{ik}\right] u_s^i u_s^j u_s^k - 2\, g_{ij} u_s^j \left[\Gamma^i_{kl}\, u_s^k u_s^l + g^{ik} h^{pq} H_k^p \left(H^q_{mn} - H_l^q \Gamma^l_{mn}\right) u_s^m u_s^n\right] = 0 \tag{35}$$

or after re-labelling indices and contracting the metric tensor

$$2\, g_{il}\Gamma^l_{jk}\, u_s^i u_s^j u_s^k - 2\, g_{ij}\Gamma^i_{kl}\, u_s^k u_s^j u_s^l - 2\, h^{pq} H_k^p\, u_s^k \left(H^q_{mn} - H_l^q \Gamma^l_{mn}\right) u_s^m u_s^n = 0 \tag{36}$$

This equation is satisfied everywhere along **C** because of the symmetry of the metric tensor and the vanishing of equations (10). This shows that the derivative of equation (19) is an identity regardless of the value of the constant in the right hand side of (19). Therefore, if the initial direction $u_s^i(0)$ is chosen such that (19) is satisfied at the initial point $s = 0$, then (19) is satisfied everywhere along C.

The construction is always valid except at singular points or other loci where the normals to two or more constraint surfaces coincide. In the non-holonomic case there is an additional condition which has not been investigated.

APPENDIX III: GENERALIZED EIGENVALUE PROBLEM WITH CONSTRAINTS

The generalized eigenvalue problem is concerned with the minimization of a ratio $\boldsymbol{Q}$ of two quadratic forms with symmetric coefficient matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.

$$\boldsymbol{Q} = \frac{\boldsymbol{x}^T \boldsymbol{A}\, \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{B}\, \boldsymbol{x}} \tag{1}$$

It is often assumed that the matrix B which plays the role of a metric is positive definite but this s not necessary. The solution of the generalized eigenvalue problem is usually carried out by performing a decomposition on the matrix B which is then used in a linear transformation which is such that the transformed B becomes the identity matrix. The Cholesky decomposition is normally used since it is cheaper to compute than the Schur decomposition. I prefer to use the Schur decomposition since it requires no inversion of the triangular Cholesky factors [which I don't have in my library] and since it can also solve the problem of indefinite B, therefore

$$\boldsymbol{B} = \boldsymbol{U}^T \Lambda\, \boldsymbol{U} \tag{2}$$

Substituting

$$\boldsymbol{y} = \Lambda^{\frac{1}{2}} \boldsymbol{U} \boldsymbol{x} \tag{3}$$

brings the matrix into the form fro the usual eigenvalue problem

$$\boldsymbol{Q} = \frac{\boldsymbol{y}^T [\Lambda^{-\frac{1}{2}} \boldsymbol{U} \boldsymbol{A}\, \boldsymbol{U}^T \Lambda^{-\frac{1}{2}}] \boldsymbol{y}}{\boldsymbol{y}^T \boldsymbol{y}} \tag{4}$$

Minimization of the ratio of quadratic forms with B a unit matrix is equivalent to the solution of the eigenvalue problem for a symmetric matrix. The solution of the eigenvalue problem may be obtained using the QR method or using the new "Divide and Conquer" method.

Now consider the problem of a biquadratic form of order n subject to a set of m linear homogeneous constraints

$$\boldsymbol{C} \boldsymbol{x} = \boldsymbol{0} \tag{5}$$

The homogeneous constraints are planes passing through the origin and the rows of C are the normals to those planes. This problem is solved by computing the null space of the matrix **C** using the SVD. The null space is the orthogonal complement of the singular vectors with nonzero singular value. The quadratic form is projected onto a basis for the null space and its eigenvectors and eigenvalues found in the reduced space. Then the eigenvectors are transformed back to the full space. In the case of the constrained generalized eigenvalue problem this step should precede the solution of the generalized eigenvalue problem. It obviously is no problem for this algorithm if the constraint matrix is not of full rank since in that case the basis for the null space will be reduced accordingly.

It is interesting to investigate the change in the invariants of the matrix of the quadratic form under this projection. According to Gantmacher[1], Kronecker has shown that the ordered eigenvalues of a singly reduced quadratic form interlace with those of the original biquadratic form. I would like to extend this analysis to changes in the invariants due to the constraints.

Let us start by considering the case of *three-dimensional* space with a *single* constraint

$$c_k x^k = 0 \tag{6}$$

This case is most easily solved using the Householder matrix. The Householder Matrix **H** based on a vector **u** reverses the sign of **u** and leaves the orthogonal complement of space intact

$$H = I - 2 u u^T \tag{7}$$

To perform a rotation of the normal to the constraint plane into the positive **z** -axis we use the vector

$$u = [0,0,1]^T - c \tag{8}$$

The quadratic form is transformed using the Householder matrix and the transformation is followed by the removal of the third dimension which is equivalent to taking minors in the matrices of the quadratic form. This leaves us with the reduced quadratic form. After calculating its eigenvectors we can transform back to the original space.

Denoting the reduced quadratic form with Q^* we find that the trace of the original biquadratic form and the trace of the reduced form are related as

$$tr(Q) - tr(Q^*) = \frac{(g^{km} g^{ln} a_{mn}) c_k c_l}{g^{kl} c_k c_l} \tag{9}$$

whereas the determinant of the original form and the determinant of the reduced form are related as

$$\frac{\det(Q^*)}{\det(Q)} = \frac{a^{-1}_{kl} c_k c_l}{g^{kl} c_k c_l} \tag{10}$$

These results for a three-dimensional space with a single constraint are of course quite ad-hoc but they are interesting nonetheless.

In chapter 2 we showed in a different context that in the case of a single constraint [H=0] [the case of a hyperspace] the principal directions and principal values of the constraint manifold are the extremal directions and extremal values of the ratio of quadratic forms

$$Q = \frac{H_{kl} p^k p^l}{g_{kl} p^k p^l} \tag{11}$$

subject to the single [tangent plane] constraint

$$H_k p^k = 0 \tag{12}$$

The Gaussian curvature $\boldsymbol{K}$ [the product of the principal curvatures] is equal to the determinant of the reduced bi-quadratic form which is found as

$$\boldsymbol{K} = \frac{1}{(H_x^2+H_y^2+H_z^2)^2}\left[\left\{\begin{matrix} H_x^2(H_{yy}H_{zz}-H_{yz}^2) \\ +H_y^2(H_{zz}H_{xx}-H_{zx}^2) \\ +H_z^2(H_{xx}H_{yy}-H_{xy}^2) \end{matrix}\right\} + 2\left\{\begin{matrix} H_xH_y(H_{xz}H_{yz}-H_{xy}H_{zz}) \\ +H_yH_z(H_{yx}H_{zx}-H_{yz}H_{xx}) \\ +H_zH_x(H_{zy}H_{xy}-H_{zx}H_{yy}) \end{matrix}\right\}\right] \tag{13}$$

The mean curvature $\boldsymbol{Km}$ [the sum of the principal curvatures] is equal to the trace of the reduced biquadratic form

$$\boldsymbol{Km} = \frac{1}{(H_x^2+H_y^2+H_z^2)^{3/2}}\left[\left\{\begin{matrix} H_x^2(H_{yy}+H_{zz}) \\ +H_y^2(H_{zz}+H_{xx}) \\ +H_z^2(H_{xx}+H_{yy}) \end{matrix}\right\} - 2\left\{\begin{matrix} H_xH_yH_{xy} \\ +H_yH_zH_{yz} \\ +H_zH_xH_{zx} \end{matrix}\right\}\right] \tag{14}$$

For a *minimal surface* we have $\boldsymbol{Km=0}$ so that the equation of a minimal surface is from (12)

$$\begin{aligned} &H_x^2(H_{yy}+H_{zz}) + H_y^2(H_{zz}+H_{xx}) + H_z^2(H_{xx}+H_{yy}) \\ &\quad - 2(H_xH_yH_{xy} + H_yH_zH_{yz} + H_zH_xH_{zx}) = 0 \end{aligned} \tag{15}$$

The same result is obtained using methods of classical differential geometry constructing a metric tensor for the implicitly given surface $\boldsymbol{H}$. For example one can choose as the coordinates to describe the surface the Cartesian coordinates [x,y] coordinates. This computation is performed in Appendix V.

APPENDIX IV: CLASSICAL TREATMENT OF A SURFACE IN THREE-DIMENSIONAL SPACE

A simple illustration of the foregoing is the computation of the Gaussian curvature of a surface in three-dimensional Cartesian space given in implicit form as

$$H(x,y,z) = 0 \tag{1}$$

First, we compute the fundamental tensors the traditional way, by constructing a metric for the surface based on a parameterization of the surface - in this case using the coordinates x and y.

In this case the covariant representation of the metric tensor is found as

$$g_{ij} = \frac{1}{H_z^2}\begin{bmatrix} H_x^2 + H_z^2 & H_x H_y \\ H_x H_y & H_y^2 + H_z^2 \end{bmatrix} \tag{2}$$

and its matrix inverse, the contravariant representation is

$$g^{ij} = \frac{1}{H_x^2 + H_y^2 + H_z^2}\begin{bmatrix} H_y^2 + H_z^2 & -H_x H_y \\ -H_x H_y & H_x^2 + H_z^2 \end{bmatrix} \tag{3}$$

and the determinant of the metric tensor

$$g = \frac{H_x^2 + H_y^2 + H_z^2}{H_z^2} \tag{4}$$

The six Christoffel symbols are computed as

$$\Gamma^i_{jk} = \frac{H_i}{H_z^2\,(H_x^2 + H_y^2 + H_z^2)}\left[\, H_z^2 H_{jk} - H_z\,(H_j H_{kz} + H_k H_{jz}) + H_j H_k H_{zz} \,\right] \tag{5}$$

The three components of the unit normal are

$$X^i = \frac{1}{(H_x^2 + H_y^2 + H_z^2)^{1/2}}\left[\, H_x \; H_y \; H_z \,\right] \tag{6}$$

The second fundamental tensor for the surface is found as

$$d_{ij} = \frac{1}{H_z^3} \begin{bmatrix} H_x^2 H_{zz} + H_z^2 H_{xx} - 2 H_x H_z H_{xz} & H_z^2 H_{xy} + H_x H_y H_{zz} - H_x H_z H_{yz} - H_y H_z H_{xz} \\ H_z^2 H_{xy} + H_x H_y H_{zz} - H_x H_z H_{yz} - H_y H_z H_{xz} & H_y^2 H_{zz} + H_z^2 H_{yy} - 2 H_y H_z H_{yz} \end{bmatrix} \quad (7)$$

In the above equations, the indices {i,j,k} assume the values 1 and 2.

This completes the computation of the curvature of the surface using the traditional method which requires the introduction of a two-parameter parameterization of the surface - for which we have used the variables x and y. matrix (7) reduces to the scalar

$$h = H_k H_k = H_x^2 + H_y^2 + H_z^2 \quad (1)$$

APPENDIX V: ELLIPSOID EXAMPLE

It is instructive to compare the results obtained using the constrained geodesic with the classical by calculating some simple examples For example, for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{2}$$

The geodesics on the ellipsoid are from the constrained geodesic equation and (9)

$$\begin{bmatrix} x_{ss} \\ y_{ss} \\ z_{ss} \end{bmatrix} = - \frac{\left\{\frac{x_s}{a}\right\}^2 + \left\{\frac{y_s}{b}\right\}^2 + \left\{\frac{z_s}{c}\right\}^2}{\left\{\frac{x}{a^2}\right\}^2 + \left\{\frac{y}{b^2}\right\}^2 + \left\{\frac{z}{c^2}\right\}^2} \begin{bmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \\ \frac{z}{c^2} \end{bmatrix} \tag{3}$$

subject to the tangent plane condition

$$\frac{x}{a^2} x_s + \frac{y}{b^2} y_s + \frac{z}{c^2} z_s = 0 \tag{4}$$

and the unit norm condition

$$x_s^2 + y_s^2 + z_s^2 = 1 \tag{5}$$

The curvature of the geodesic is

$$\kappa = \frac{\left\{\frac{x_s}{a}\right\}^2 + \left\{\frac{y_s}{b}\right\}^2 + \left\{\frac{z_s}{c}\right\}^2}{\sqrt{\left\{\frac{x}{a^2}\right\}^2 + \left\{\frac{y}{b^2}\right\}^2 + \left\{\frac{z}{c^2}\right\}^2}} \tag{6}$$

For example, at the point u = [a,0,0] we obtain for the initial direction us = [0,1,0]: kappa = a/b^2 and for the initial direction us=[0,0,1]: kappa = a/c^2

The Gaussian curvature of the surface can also be obtained from our theory

$$K = \frac{1}{a^2 b^2 c^2 \left[\left\{\frac{x}{a^2}\right\}^2 + \left\{\frac{y}{b^2}\right\}^2 + \left\{\frac{z}{c^2}\right\}^2 \right]^2} \tag{7}$$

The mean curvature of the surface does not follow from the theory but is known as

$$K_m = \frac{(b^2+c^2)\dfrac{x^2}{a^2} + (c^2+a^2)\dfrac{y^2}{b^2} + (a^2+b^2)\dfrac{z^2}{c^2}}{a^2 b^2 c^2 \left[\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4}\right]^{\frac{3}{2}}} \tag{8}$$

For example, at the point [a,0,0] we find for the Gaussian and mean curvature of the surface

$$K = \left[\frac{a}{b\,c}\right]^2$$
$$K_m = a\left[\frac{1}{b^2} + \frac{1}{c^2}\right] \tag{9}$$

from which we can deduce that the principal curvatures are

$$\kappa_1 = \frac{a}{b^2}$$
$$\kappa_2 = \frac{a}{c^2} \tag{10}$$

These results can also be obtained form classical surface theory. It would involve choosing a parameterization of the ellipsoid [x,y] and computing the Christoffel symbols for the surface. Note that the third geodesic equation would have to be inferred from the first two which in this case is easy by symmetry. The classical theory would only give two of the three geodesic equations.

The classical theory also allows the natural parameterization similar to the definition of spherical polar coordinates

$$x = a\cos(\phi)\sin(\theta)$$
$$y = b\sin(\phi)\sin(\theta) \tag{11}$$
$$z = c\cos(\theta)$$

In simple geometries such as the above, the advantage of the constrained geodesic theory is that it gives the results expressed in the coordinates of the enveloping space. The negative side is that the geodesics cannot be integrated without exactly honouring the constraints where for surfaces with a known parameterization this is possible. Note that there too the geodesic is not exact due to numerical integration errors. In more complicated situations such as multiple constraints finding a parameterization of a surface may bot be possible.

APPENDIX VI: NOTE ON THE GEODESIC SERIES

It is possible to eliminate any explicit dependency on α and write the derivative α' explicitly

$$[H_k^p H_k^u]\,[(H_{kl}^u H_k^v - H_{kl}^u H_k^v)\,u_s^l]^{-1}\,[H_k^v H_k^q]\,\alpha_q' = -[H_{kl}^p u_s^k u_s^l] - [H_k^p H_k^u]\,[(H_{kl}^u H_k^v - H_{kl}^u H_k^v)\,u_s^l]^{-1}\,H_{klm}^v u_s^k u_s^l u_s^m \tag{1}$$

provided the indicated inverse matrix exists. In the special case of quadratic constraints this simplifies further to

$$[H_k^p H_k^u]\,[(H_{kl}^u H_k^v - H_{kl}^u H_k^v)\,u_s^l]^{-1}\,[H_k^v H_k^q]\,\alpha_q' = -[H_{kl}^p u_s^k u_s^l] \tag{2}$$

Let us start with quadratic constraints

$$[H_k^p H_k^u]\,[(H_{kl}^u H_k^v - H_{kl}^u H_k^v)\,u_s^l]^{-1}\,[H_k^v H_k^q]\,\alpha_q' = \beta_p \tag{3}$$

Introducing the matrices

$$g_{pq} = [(H_{kl}^p H_k^q - H_{kl}^q H_k^p)\,u_s^l] \tag{4}$$

$$h_{pq} = [H_k^p H_k^q] \tag{5}$$

we may write this as

$$[h\, g^{-1} h]\,\alpha' = \beta \tag{6}$$

Differentiating a second time

$$[\,[h'\,(g^{-1}h)] + [(h\,g^{-1})\,h'] - [h\,g^{-1} g'\,g^{-1} h]\,]\,\alpha' + [h\,g^{-1} h]\,\alpha'' = 2\,H_{kl}^p u_{ss}^k u_s^l \tag{7}$$

Substituting for u_{ss} from (20)

$$[\,[h'(g^{-1}h)] + [(hg^{-1})\,h'] - [h\,g^{-1} g'\,g^{-1} h]\,]\,\alpha' + [h\,g^{-1} h]\,\alpha'' = -2\,H_{kl}^p H_k^q\,\alpha_q u_s^l \tag{8}$$

$$[h\,g^{-1} h]\,\alpha' + [h\,g^{-1} h]\Big[\,[h'(g^{-1}h)] + [(hg^{-1})\,h'] - [h\,g^{-1} g'\,g^{-1} h]\,\Big]^{-1}[h\,g^{-1} h]\,\alpha'' = -2\,[h\,g^{-1} h]\Big[\,[h'(g^{-1}h)] + [(hg^{-1})\,h'] - [h\,g^{-1} g'\,g^{-1} h]\,\Big]^{-1} H_{kl}^p H_k^q\,\alpha_q u_s^l \tag{9}$$

$$[h\ g^{-1}h]\left[[h'(g^{-1}h)] + [(hg^{-1})h'] - [h g^{-1} g' g^{-1} h]\right]^{-1}[h g^{-1} h]\,\alpha'' = -[h\ g^{-1}h]\,\alpha' - 2[h\ g^{-1}h]\left[[h'(g^{-1}h)] + [(hg^{-1})h'] - [h g^{-1} g' g^{-1} h]\right]^{-1} H^p_{kl} H^q_k\, \alpha_q\, u^l_s \tag{10}$$

In summary

$$h_{pq} = [H^p_k H^q_k] \tag{11}$$

$$g_{pq} = [(H^p_{kl} H^q_k - H^q_{kl} H^p_k)\, u^l_s] \tag{12}$$

Impossible, g is singular

$$M = [h\ g^{-1}h] \tag{13}$$

$$N = M' = [h'(g^{-1}h)] + [(hg^{-1})h'] + [(hg^{-1})g'(g^{-1}h)] \tag{14}$$

$$[M]\ \alpha' = \beta \tag{15}$$

$$[MN^{-1}M]\ \alpha'' = -\beta - 2MN^{-1}[H^p_{kl} H^q_k\, \alpha_q\, u^l_s] \tag{16}$$

$$[M]\ \alpha'' = -NM^{-1}\beta - 2[H^p_{kl} H^q_k\, u^l_s]\,\alpha_q \tag{17}$$

$$K_{pq} = [H^p_{kl} H^q_k\, u^l_s] \tag{18}$$

For the sphere problem K is not invertible in fact K[n,n] is zero and only K[n+1:n+3][n+1:n+3] is nonzero.

$$[K^{-1}M]\ \alpha'' = -K^{-1}NM^{-1}\beta - 2\,\alpha_p \tag{19}$$

$$[h\ K^{-1}M]\ \alpha'' = -hK^{-1}NM^{-1}\beta - 2\,h\,\alpha_q \tag{20}$$

$$[h\ K^{-1}M]\ \alpha'' = -hK^{-1}NM^{-1}\beta - 2\,\beta \tag{21}$$

$$[h\ K^{-1}M]\ \alpha'' = -[2I + hK^{-1}NM^{-1}]\,\beta \tag{22}$$

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